Section 4.3

Linear, homogeneous equations with constant coefficients

There are three possible types of solutions for the homogeneous equation of this type.

The substitution: $y = e^{mx}$ $\qquad y_c = c_1 y_1 + c_2 y_2 + \cdots$

type 1 distinct roots

Ex $\quad m_1 = 5 \quad m_2 = 8 \qquad y_c = c_1 e^{5x} + c_2 e^{8x}$

type 2 repeating roots

Ex $\quad m_1 = 4 \quad m_2 = 4 \qquad y_c = c_1 e^{4x} + c_2 x e^{4x}$

type 3 complex roots

Ex $\quad m = 4 \pm \sqrt{3}\, i \qquad$ so $\alpha = 4$

$m = \alpha \pm \beta i \qquad \beta = \sqrt{3}$

form: $y_c = e^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right]$

$$\boxed{y_c = e^{4x}\left[c_1 \cos \sqrt{3}\, x + c_2 \sin \sqrt{3}\, x\right]}$$

Example 1 Find the general solution to the homogeneous equation

$$5y^{(4)} + 3y^{(3)} = 0$$

$5m^4 + 3m^3 = 0$

$m^3[5m+1] = 0$

$m^3 = 0$

(repeated 3 times)

$5m+1=0$

$5m=-1$

$m=-\frac{1}{3}$

$$y = c_1 e^0 + c_2 x e^0 + c_3 x^2 e^0 + c_4 e^{-\frac{1}{3}x}$$

$$\boxed{y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-\frac{1}{3}x}}$$

Example 2 $y^{(4)} + 2y^{(3)} + 3y'' + 2y' + y = 0$

Find the general solution

$m^4 + 2m^3 + 3m^2 + 2m + 1 = 0$

possible real roots $\rightarrow m = \pm 1$ $\quad m(1) = 9$

$m(-1) = 1$

This means no real roots.

So now we look for a trinomial squared. The first and last coefficients must be 1, so let's try $(m^2+m+1)^2$

$$\begin{aligned}(m^2+m+1)(m^2+m+1) = \; & m^4 + m^3 + m^2 \\ & + m^3 + m^2 + m \\ & + m^2 + m + 1 \\ \hline & m^4 + 2m^3 + 3m^2 + 2m + 1\end{aligned}$$

so $m^4 + m^3 - 3m^2 - 5m - 2 = 0$

$(m+1)(m-2)(m^2+2m+1) = 0$

$(m+1)(m-2)(m+1)(m+1) = 0$

$(m+1)^3(m-2) = 0$

$m = -1$ (repeated 3 times) $\quad m = 2$

$$\boxed{y_c = y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + c_4 e^{2x}}$$

Example 4

Solve the initial value problem

$2y''' - 3y'' - 2y' = 0 \qquad y(0) = 1$

$y'(0) = -1$

$y''(0) = 3$

auxiliary equation

$2m^3 - 3m^2 - 2m = 0$

$m(2m^2 - 3m - 2) = 0$

$m(2m+1)(m-2) = 0$

$m_1 = 0 \quad m_2 = -\frac{1}{2} \quad m = 2$

$y = c_1 e^0 + c_2 e^{-\frac{1}{2}x} + c_3 e^{2x}$

$$\boxed{y = c_1 + c_2 e^{-\frac{1}{2}x} + c_3 e^{2x}}$$

$y' = -\frac{1}{2} c_2 e^{-\frac{1}{2}x} + 2c_3 e^{2x}$

$y'' = \frac{1}{4} c_2 e^{-\frac{1}{2}x} + 4c_3 e^{2x}$

now find c_1, c_2, c_3

$y(0) = c_1 + c_2 + c_3 = 1$

$y'(0) = -\frac{1}{2}c_2 + 2c_3 = -$

$y''(0) = \frac{1}{4}c_2 + 4c_3 = 3$

$\frac{1}{4}c_2 + 4c_3 = 3$

$\boxed{c_2 + 16c_3 = 12}$

$-\frac{1}{2}c_2 + 2c_3 = -1$

$\boxed{c_2 - 4c_3 = 2}$

Extra Practice Problems for Section 4.3

Find the general solution for each of the following differential equations.

1. $y'' - 4y = 0$
2. $y'' + 3y' - 10y = 0$
3. $y'' + 6y' + 9y = 0$
4. $4y'' - 12y' + 9y = 0$
5. $y'' + 8y' + 25y = 0$
6. $y^{(4)} - 8y''' + 16y'' = 0$
7. $y^{(4)} - 3y^{(3)} + 3y'' - y' = 0$
8. $9y''' + 12y'' + 4y' = 0$
9. $y^{(4)} - 8y'' + 16y = 0$
10. $6y^{(4)} + 11y'' + 4y = 0$
11. $y''' + y'' - y' - y = 0$

Solve each Initial Value Problem

12. $y'' - 4y' + 3y = 0 \quad y(0) = 7,\ y'(0) = 11$
13. $y'' - 6y' + 4y = 0 \quad y(0) = 3,\ y'(0) = 1$
14. $3y''' + 2y'' = 0 \quad y(0) = -1,\ y'(0) = 0,\ y''(0) = 1$

Find the general solution of each differential equation

(15) $y''' + 3y'' - 4y = 0$

(16) $y''' + 27y = 0$

(17) $y''' + 3y'' + 4y' - 8y = 0$

(18) $y''' + 3y'' - 54y = 0$

(19) $6y^{(4)} + 5y''' + 25y'' + 20y' + 4y = 0$

[Hint: one solution is $y = \cos 2x$]

Answers

(1) $y = c_1 e^{2x} + c_2 e^{-2x}$

(2) $y = c_1 e^{2x} + c_2 e^{-5x}$

(3) $y = c_1 e^{-3x} + c_2 x e^{-3x}$

(4) $y = c_1 e^{\frac{3}{2}x} + c_2 x e^{\frac{3}{2}x}$

(5) $y = e^{-4x}[c_1 \cos 3x + c_2 \sin 3x]$

(6) $y = c_1 + c_2 x + c_3 e^{4x} + c_4 x e^{4x}$

(7) $y = c_1 + c_2 e^{x} + c_3 x e^{x} + c_4 x^2 e^{x}$

(8) $y = c_1 + c_2 e^{-\frac{2}{3}x} + c_3 x e^{-\frac{2}{3}x}$

(9) $y = c_1 e^{2x} + c_2 x e^{2x} + 3_3 e^{-2x} + c_4 x e^{-2x}$

(10) $y = c_1 \cos\left(\frac{1}{\sqrt{2}}x\right) + c_2 \sin\left(\frac{1}{\sqrt{2}}x\right) + c_3 \cos\left(\frac{2}{\sqrt{3}}x\right)$
$+ c_4 \sin\left(\frac{2}{\sqrt{3}}x\right)$

(11) $y = c_1 e^{x} + c_2 e^{-x} + c_3 x e^{-x}$

(12) $y = 5e^{x} + 2e^{3x}$

(13) $y = e^{3x}\left[3\cos 4x - 2\sin 4x\right]$

(14) $y = \frac{1}{4}\left(-13 + 6x + 9e^{-\frac{2}{3}x}\right)$

(15) $y = c_1 e^{x} + c_2 e^{-2x} + c_3 x e^{-2x}$

(16) $y = c_1 e^{-3x} + e^{-\frac{3}{2}x}\left[c_2 \cos\left(\frac{3\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{3\sqrt{3}}{2}x\right)\right]$

(17) $y = c_1 e^{x} + e^{-2x}\left(c_2 \cos 2x + c_3 \sin 2x\right)$

(18) $y = c_1 e^{-\frac{1}{2}x} + c_2 e^{-\frac{1}{3}x} + c_3 \cos 2x + c_4 \sin 2x$